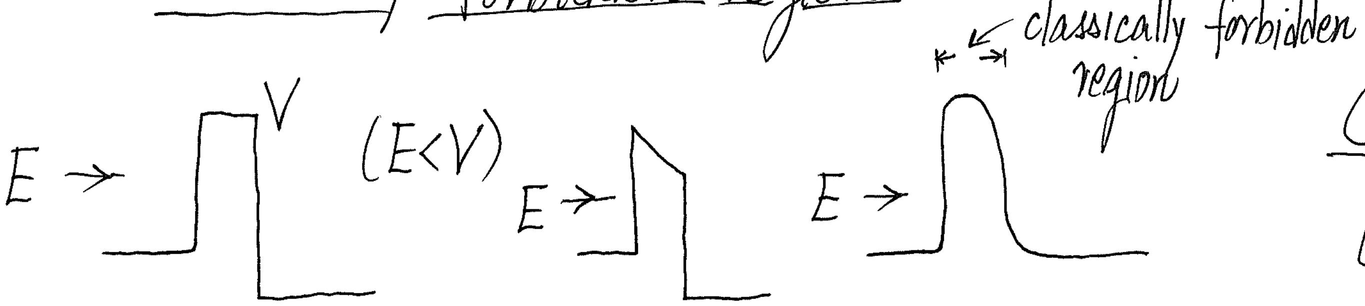
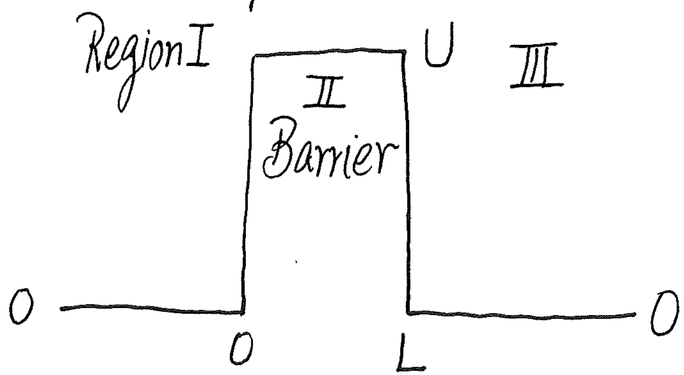
C. Tunneling : Simplest case

- Tunneling refers to a finite probability of getting through a classically forbidden region



Classically, particle cannot be found at places with $E<V$

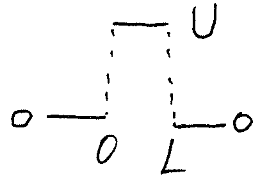
Simplest Case (1D square barrier)



E = energy of incident particles
($E<U$)

[Can easily be extended to $E>U$]

- "Simple" because $V(x)$ is <u>piecewise constant</u>
  and $V=0$ (same) on both sides



<u>Write down wavefunctions in each region</u>

<u>Region I</u>: $(x<0)$

$$\psi_{I}(x) = \underbrace{A\,e^{ikx}}_{\substack{\text{incident}\\ \text{(towards right)}}} + \underbrace{B\,e^{-ikx}}_{\substack{\text{reflected}\\ \text{(towards left)}}}, \quad x<0 \qquad (17)$$

$\because V=0$, $\dfrac{\hbar^2 k^2}{2m} = E \Rightarrow k = \sqrt{\dfrac{2mE}{\hbar^2}}$ (18)

[Where is time? Think of it as a steady state situation (or $e^{-i\omega t}$ in time)], where $\omega = E/\hbar$

<u>Region III</u>: $(x>L)$

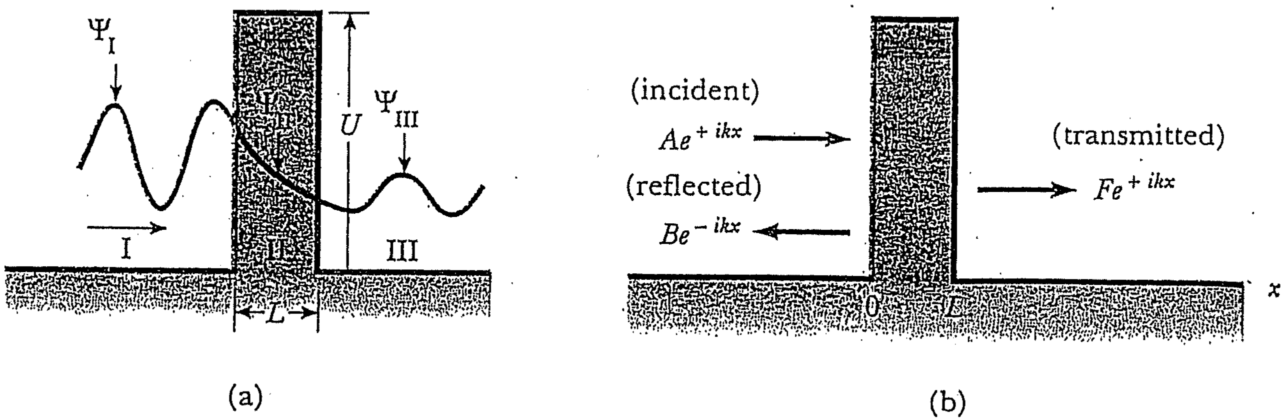
$$\psi_{III}(x) = \underbrace{F\,e^{ikx}}_{\substack{\text{transmitted}\\ \text{(towards right)}}}, \quad x>L\,; \quad k = \sqrt{\frac{2mE}{\hbar^2}} \quad (\because V=0) \qquad (19)$$

Region II: $(0<x<L)$

$$\psi_{II}(x) = \underbrace{C e^{-Kx}}_{\text{dropping}} + \underbrace{D e^{+Kx}}_{\text{growing}}, \quad 0<x<L \quad (20)$$

because barrier width is finite (keep both)

$$K = \sqrt{\frac{2m}{\hbar^2}(U-E)} \quad (21) \quad (\underbrace{U>E}_{\text{tunneling}})$$



(a) A typical stationary state wave for a particle in the presence of a square barrier. The energy E of the particle is less than the barrier height U. Since the wave amplitude is nonzero in the barrier, there is some probability of finding the particle there. (b) Decomposition of the stationary wave into incident, reflected, and transmitted waves.

What do we want to obtain?

- Invoke prob. current density J (Eq. (14a))

(i) Region I: $(x<0)$

$$J = \underbrace{\frac{\hbar k}{m}|A|^2}_{\text{incident}} - \underbrace{\frac{\hbar k}{m}|B|^2}_{\text{reflected}} \quad (\text{in } \hat{x}) \quad (\text{see Eq. (16)})$$

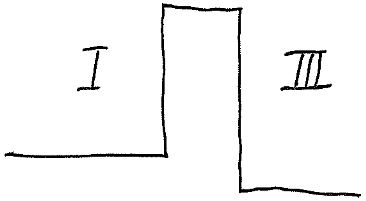
Reflection Coefficient $R \underset{\text{[general]}}{\equiv} \dfrac{|\text{Reflected current density}|}{\text{Incident current density}} = \dfrac{\frac{\hbar k}{m}|B|^2}{\frac{\hbar k}{m}|A|^2} = \dfrac{|B|^2}{|A|^2}$ (22)

(ii) Region III: $(x>L)$

$$J = \frac{\hbar k}{m}|F|^2 \quad (\text{see Eq. (15)})$$

Transmission Coefficient $T \underset{\text{[general]}}{\equiv} \dfrac{\text{Transmitted current density}}{\text{Incident current density}} = \dfrac{\frac{\hbar k}{m}|F|^2}{\frac{\hbar k}{m}|A|^2} \overset{\text{for } \sqcap}{=} \dfrac{|F|^2}{|A|^2}$ (23)

Want to get R and T as function of E. (given L, U characterizing barrier)

Note: For 

, $T \neq \dfrac{|F|^2}{|A|^2}$

$\because \; k_{I} \neq k_{III}$ (same E)

$$T = \frac{\text{Transmitted current density}}{\text{Incident current density}} \quad (\text{Def.})$$

$$= \frac{k_{III}}{k_{I}} \cdot \frac{|F|^2}{|A|^2}$$

This is why we need the concept of $\vec{J}$ to do tunneling problems.

- Quick accounting: A, B, C, D, F in ψ_I, ψ_{II}, ψ_{III}

Two Boundaries at $\underbrace{x=0}$ and $\underbrace{x=L}$ $\Rightarrow$ 4 equations

[ψ continuous and $\frac{d\psi}{dx}$ continuous]† boundary conditions

Looks like 4 equations for 5 unknowns !?

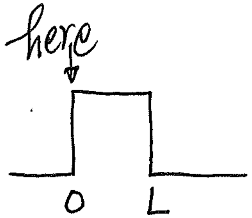
But we only need R and T, thus only ratios. ($\therefore$ OK!)

- Can set $A=1$ at the beginning, i.e. $\psi_I = e^{ikx} + Be^{-ikx}$ $(x<0)$ and work out B, C, D, F (particularly B & F)
- Or work out B/A, C/A, D/A, F/A.

So, set $A=1$

$\dagger$ The boundary conditions also make $\vec{J}$ continuous (no source)

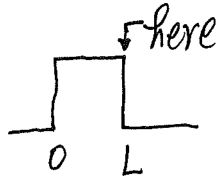
- Interface (Boundary) at $x=0$



ψ continuous $\Rightarrow$ $\quad 1 + B = C + D$

$\dfrac{d\psi}{dx}$ continuous $\Rightarrow$ $\quad ik - ikB = -\kappa C + \kappa D$

- Interface (Boundary) at $x = L$



ψ continuous $\Rightarrow$ $\quad C e^{-\kappa L} + D e^{+\kappa L} = F e^{ikL}$

$\dfrac{d\psi}{dx}$ continuous $\Rightarrow$ $\quad -\kappa C e^{-\kappa L} + \kappa D e^{+\kappa L} = ik F e^{ikL}$

$\boxed{\text{Four equations to solve for } B, C, D, F}$ (Ex.) [All physics is used!]

<u>Note</u>: <u>System's parameters</u> are hidden in k and κ (and L appears in eqs.)

- barrier height U
- Incident energy E

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \kappa = \sqrt{\frac{2m(U-E)}{\hbar^2}}$$

- Key Point is that $F \neq 0$ (despite tiny) even for $E<U$

$$F = \frac{2i\kappa k}{2i\kappa k\cosh(\kappa L) + (k^2-\kappa^2)\sinh(\kappa L)} \quad (24) \quad (\text{Ex.})$$

$$\therefore T(E) = |F|^2 = \frac{4\kappa^2 k^2}{4\kappa^2 k^2\cosh^2(\kappa L) + (k^2-\kappa^2)^2\sinh^2(\kappa L)} = \frac{4\kappa^2 k^2}{4\kappa^2 k^2 + (k^2+\kappa^2)^2\sinh^2(\kappa L)}$$

Transmission coefficient

$$= \frac{1}{1 + \frac{1}{4}\cdot\frac{U^2}{E(U-E)}\cdot\sinh^2\left(\sqrt{\frac{2m}{\hbar^2}(U-E)}\cdot L\right)} \quad (25)$$

$$R(E) = |B|^2 = \frac{(\kappa^2+k^2)^2\sinh^2(\kappa L)}{4\kappa^2 k^2 + (k^2+\kappa^2)^2\sinh^2(\kappa L)} \quad (26) \quad (\text{Ex.})$$

Reflection coefficient

$$T + R = 1$$ (Ex.) (Why? What is the physics?)

Key Quantum Sense on Tunneling

Typically, $\kappa L = \sqrt{\frac{2m(U-E)}{\hbar^2}} \cdot L \gg 1$ $\left\{ \begin{array}{l} E \ll U \\ \text{(low incident energy) (high barrier)} \\ \text{and/OR} \\ \text{wide barrier} \end{array} \right.$

[thus $\sinh(\kappa L) = \frac{e^{\kappa L} - e^{-\kappa L}}{2} \approx \frac{e^{\kappa L}}{2}$]

From Eq.(25) for $T(E)$:

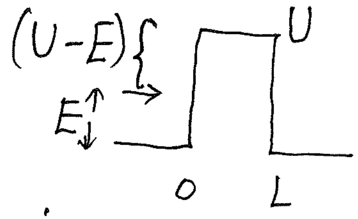
$$\boxed{T \approx 16\left(\frac{E}{U}\right)\left(1-\frac{E}{U}\right) \underbrace{e^{-2L\sqrt{\frac{2m}{\hbar^2}(U-E)}}}_{}} \quad (27)$$

Key Result!
[More important than derivation!]

Very sensitive dependence on L, m, $(U-E)$
because of the exponential function

Ex: Plug in numbers and get a sense of how small $T(E)$ could be.

L = Width of Barrier where $E < U$



$U - E$ = deficiency in energy to overcome barrier

$$T \propto e^{-2L\sqrt{\frac{2m}{\hbar^2}(U-E)}}$$

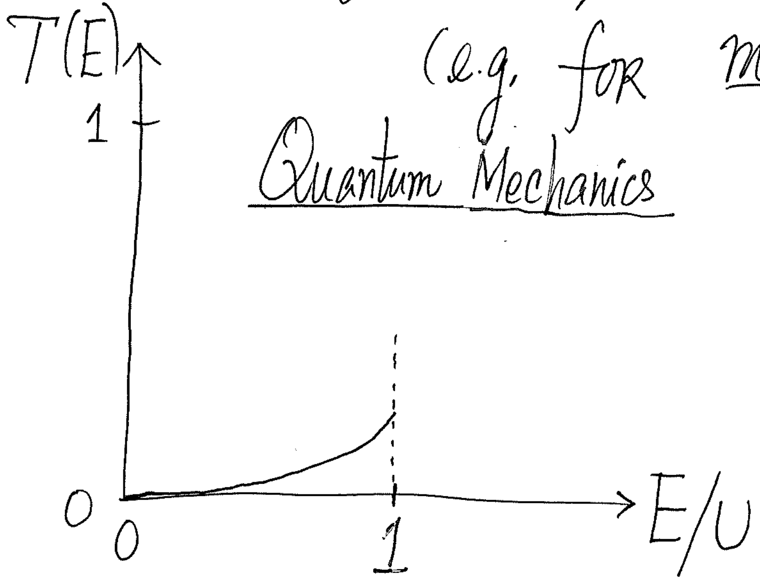
- drops a lot if L is just a bit thicker
- drops a lot if deficiency $(U-E)$ is just a bit more

Meaning: E is just a bit smaller

What does Eq. (25) say? ($E<U$)

(e.g. for $\frac{mUL^2}{\hbar^2}=8$)

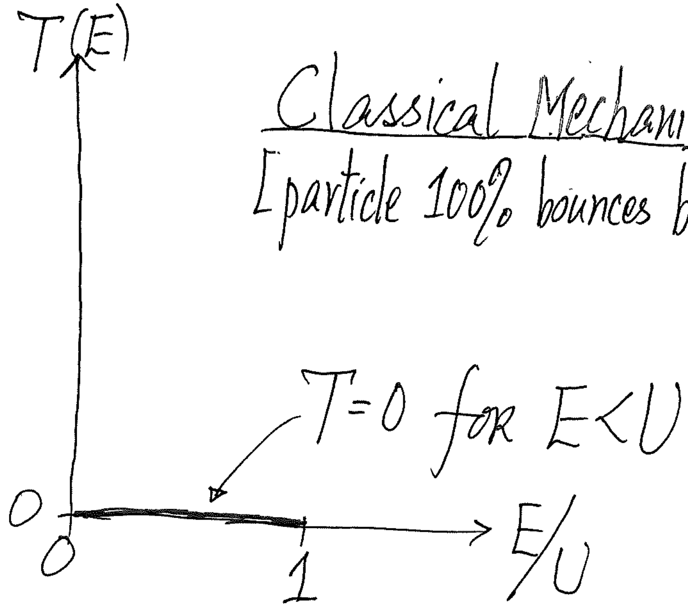
Quantum Mechanics



vs

Classical Mechanics

[particle 100% bounces back]

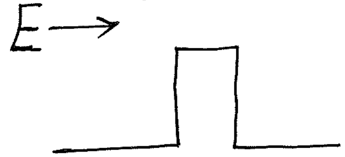


$T(E) < 1$ (for $E<U$)

$T(E)$ is small (but $\neq 0$) [tunneling]

due to wave nature (quantum mechanics)

There are Quantum Effects even for $E>U$

- $E>U$ Classical Mechanics 

$T=1$
[only slowed down in region II]

- Quantum Mechanics

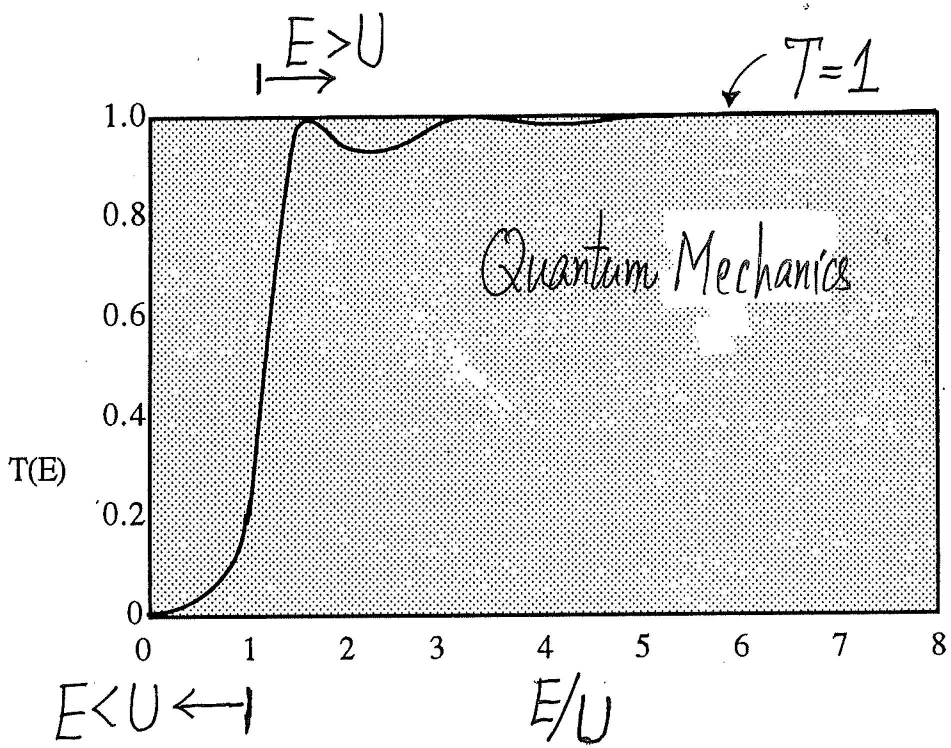
For $E>U$, $\psi_{II}(x) = C e^{ik'x} + D e^{-ik'x}$ $\quad 0<x<L$

where $k' = \sqrt{\dfrac{2m}{\hbar^2}(E-U)}$

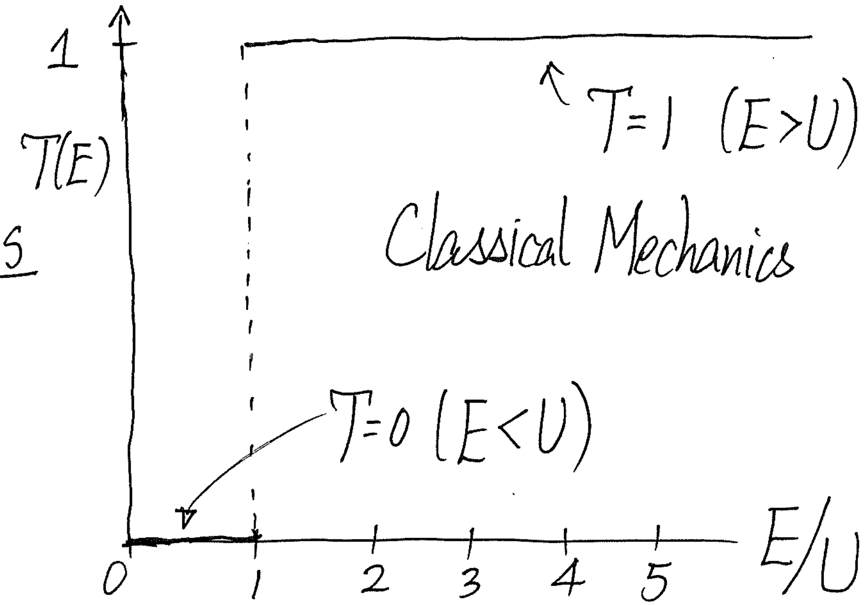
$\left[\text{c.f. } \psi_{II}(x) = C e^{-\kappa x} + D e^{\kappa x} \text{ for } E<U \right]$

$\Rightarrow$ Can simply replace κ by ik' in Eq.(25) to get $T(E)$ for $E>U$

(or match B.C.'s at $x=0$ & $x=L$ again, and repeat the derivation)



VS



There are Quantum Effects

- $T \neq 1$ (for $E \gtrsim U$)
- $T = 1$ for specific energies (transmission resonance)
- $T \to 1$ for $E \gg U$ (as in Classical Mechanics)

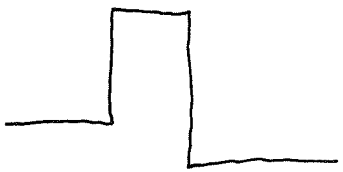
Matter Waves

$\Rightarrow$ $\boxed{\text{Physics of Waves}}$

Cross Referencing: Matter → Matter Wave [QM]

- But Waves Mathematics is not something new!
- Here is a problem in EM waves from Griffith's book
- Analogous to

E →



(*) Set $n_1 = n_3$ and compare T with $E > U$ case $(\kappa = ik')$

Problem 9.34 Light of (angular) frequency ω passes from medium 1, through a slab (thickness d) of medium 2, and into medium 3 (for instance, from water through glass into air, as in Fig. 9.27). Show that the transmission coefficient for normal incidence is given by

$$T^{-1} = \frac{1}{4n_1 n_3}\left[(n_1 + n_3)^2 + \frac{(n_1^2 - n_2^2)(n_3^2 - n_2^2)}{n_2^2}\sin^2\left(\frac{n_2 \omega d}{c}\right)\right]. \quad (*)$$

[*Hint:* To the *left*, there is an incident wave and a reflected wave; to the *right*, there is a transmitted wave; inside the slab there is a wave going to the right and a wave going to the left. Express each of these in terms of its complex amplitude, and relate the amplitudes by imposing suitable boundary conditions at the two interfaces. All three media are linear and homogeneous; assume $\mu_1 = \mu_2 = \mu_3 = \mu_0$.]

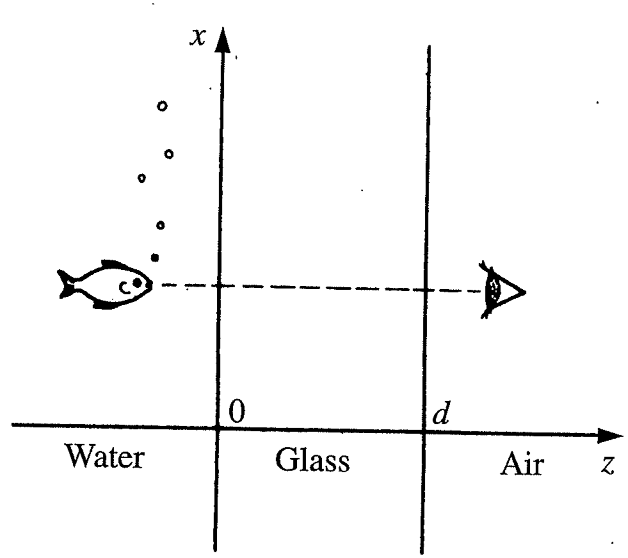
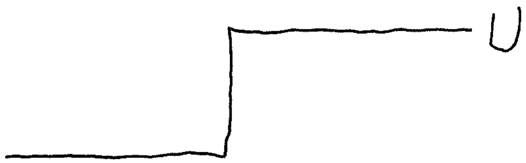


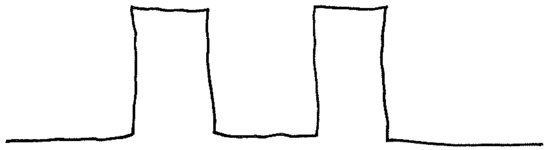
Figure 9.27

Other standard problems:

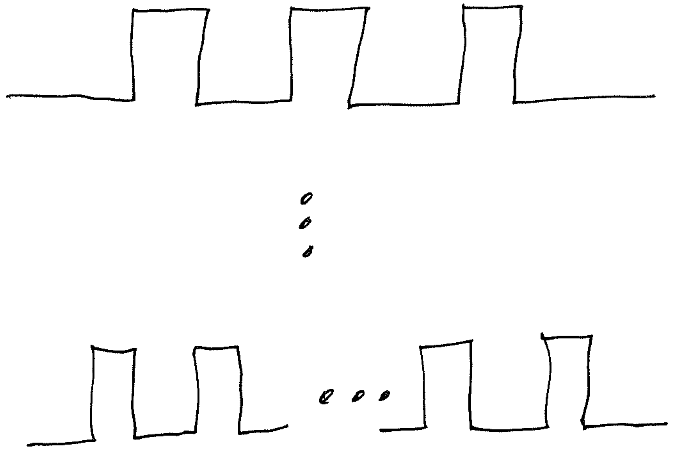


step potential

- $(E < U)$ is analogous to "copper mirror"



double barrier

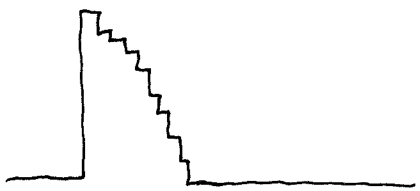


(related to solids)

Summary

- $T \propto e^{-\sqrt{\frac{2m}{\hbar^2}\underbrace{(U-E)}_{\text{energy deficiency to overcome barrier}}}\,L}$ ← barrier width (for $\kappa L \gg 1$)

  exponentially sensitive [L and/or E change by a bit, T changes drastically]

-  ? $\approx$  piecewise constant $V(x)$
  $\Rightarrow$ write ψ in each piece and match B.C.'s

Topics to learn in future courses

- WKB (Wentzel-Kramers-Brillouin) approximation
- Transfer matrix Method 

  (General situation) $[\text{out}] = \underbrace{[\,2\times 2\,]}_{\text{matrix?}}[\text{in}]$
- Numerical solutions

Go to a crash module on the Physics of Nuclei